Numerical Solution of Ordinary Differential Equations (Part - 2)

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Overview : Runge-Kutta Methods

The Taylor's series method of solving differential equations numerically is handicapped by the problem of finding the higher order derivatives.

Euler's method is less efficient in practical problems since it requires h to be small for obtaining reasonable accuracy.

The **Range-Kutta methods** do not require the calculations of higher order derivatives and they are designed to give greater accuracy with the advantage of requiring only the function values at some selected points on the sub-interval.

These methods agree with Taylor's series solution upto the terms of h^r where r is the order of the Runge-Kutta method.

In the lecture, we discuss Runge-Kutta methods of orders upto 4.

Overview : Predictor-Corrector Methods

A predictor-corrector method is an algorithm that proceeds in two steps. First, the prediction step calculates a rough approximation of the desired quantity. Second, the corrector step refines the initial approximation using another means.

This is not unlike an implementation of Newton-Raphson. In that method we require an initial guess (we "predict") and then the Newton-Raphson approach tells us how to iterate (or "correct") our latest approximation.

The main difference here is that **we have a systematic way of obtaining the initial prediction.**

We discuss some predictor-corrector methods in the lecture.

Introduction

Runge-Kutta method is a numerical technique used to solve ordinary differential equation of the form

$$\frac{dy}{dx} = f(x, y) \qquad \text{given} \quad y(x_0) = y_0.$$

Runge-Kutta methods (also called as **RK methods**) are the generalization of the concept used in Modified Euler's method. They are iterative methods to calculate the solution of a differential equation. Starting from an initial condition, they calculate the solution forward step by step.

The RK methods were first studied by the German mathematicians **Carle Runge** and **Martin Kutta** around 1900. Modern developments are mostly due to **John Butcher** in the 1960s.

The basic reasoning behind Runge-Kutta methods is outlined in the following.

In Modified Euler's method, the slope of the solution curve has been approximated with the slopes of the curve at the end points of the each sub interval in computing the solution.

The natural generalization of this concept is computing the slope by taking a **weighted average of the slopes** taken at more number of points in each sub interval. However, the implementation of the scheme differs from Modified Euler's method so that the developed algorithm is explicit in nature. The final form of the scheme is of the form

$$y_{i+1} = y_i + (\text{weighted average of the slopes}) \quad \text{for} \quad i = 0, 1, 2, \ldots$$

where h is the step length and y_i and y_{i+1} are the values of y at x_i and x_{i+1} respectively.

In general, the slope is computed at various points x in each sub interval $[x_i, x_{i+1}]$ and multiplied them with the step length h and then weighted average of it is then added to y_i to compute y_{i+1}.

Thus the RK method with r slopes called as r-**stage RK method** can be written as

$$
\begin{aligned}
K_1 &= hf(x_i, y_i) \\
K_2 &= hf(x_i + c_2 h, y_i + a_{21} K_1) \\
K_3 &= hf(x_i + c_3 h, y_i + a_{31} K_1 + a_{32} K_2) \\
\vdots &= \vdots \qquad \vdots \qquad \vdots \\
K_r &= hf(x_i + c_r h, y_i + a_{r1} K_1 + a_{r2} K_2 + \cdots + a_{rr-1} K_{r-1})
\end{aligned}
$$

and

$$
y_{i+1} = y_i + (W_1 K_1 + W_2 K_2 + \cdots + W_r K_r) \quad \text{for} \quad i = 0, 1, 2, \ldots.
$$

To determine the parameters

- c_i's for $2 \leq i \leq r$ (coefficients for x)
- a_{ij}'s for $1 \leq j < i \leq r$ (coefficients for y)
- W_i's for $1 \leq i \leq r$ (weights)

in the above equation, y_{i+1} defined in the scheme is expanded interms of step length h.

The resultant equation is then compared with Taylor series expansion of the solution of the differential equation upto a certain number of terms, say p.

Then the r-stage RK method will be of order p or is an pth order Runge-Kutta method (called Runge-Kutta method of order p).

The Butcher Tableau

To specify a particular method, we need to provide the parameters :

- r (number of stages)
- c_i's for $2 \leq i \leq r$ (coefficients for x)
- a_{ij}'s for $1 \leq j < i \leq r$ (coefficients for y)
- W_i's for $1 \leq i \leq r$ (weights).

These data are usually arranged in a co-called **Butcher tableau**.

$$
\begin{array}{c|ccccc}
0 & & & & & \\
c_2 & a_{21} & & & & \\
c_3 & a_{31} & a_{32} & & & \\
\vdots & \vdots & \vdots & \vdots & & \\
c_r & a_{r1} & a_{r2} & \dots & a_{rr-1} & \\
\hline
 & W_1 & W_2 & \dots & W_{r-1} & W_r
\end{array}
$$

Runge-Kutta Method of Order 1

Consider

$$\frac{dy}{dx} = f(x, y) \qquad \text{given} \quad y(x_0) = y_0.$$

The Euler's formula for first approximation to the solution of the above differential equation is given by

$$y_{i+1} = y_i + hf(x_i, y_i) = y_i + hy'(x_i) \quad [\text{ since } \; y' = f(x, y)\ ].$$

Also

$$y_{i+1} = y_i + hy'(x_i) + \frac{h^2}{2!}y''(x_i) + \frac{h^2}{2!}y'''(x_i) + \cdots .$$

Clearly the Euler's method agrees with the Taylor's series solution upto the term in h.

Hence Euler's method is the **Runge-Kutta method of first order.**

Runge-Kutta Method of Order 2

Now, consider the case $r = 2$ to derive the 2-stage (second order) RK method. For this

$$\begin{aligned} K_1 &= hf(x_i, y_i) \\ K_2 &= hf(x_i + c_2 h, y_i + a_{21} K_1) \\ y_{i+1} &= y_i + (W_1 K_1 + W_2 K_2) \quad \text{for} \quad i = 0, 1, 2, \ldots. \end{aligned}$$

Now by Taylor series expansion

$$\begin{aligned} y(x_{i+1}) &= y(x_i) + hy'(x_i) + \frac{h^2}{2!} y''(x_i) + \frac{h^2}{2!} y'''(x_i) + O(h^4) \\ y_{i+1} &= y_i + hf_i + \frac{h^2}{2!}(f_x + f_y f) + \frac{h^3}{3!}(f_{xx} + 2f_{xy} f + f_{yy} f^2 + f_y(f_x + f_y f)) + O(h^4) \end{aligned}$$

Runge-Kutta Method of Order 2

Also

$$
\begin{aligned}
K_1 &= hf_i \\
K_2 &= hf(x_i + c_2 h, y_i + a_{21} K_1) = hf(x_i + c_2 h, y_i + a_{21} hf_i) \\
&= h\left[f_i + c_2 h f_x + a_{21} h f_i f_y + \frac{(c_2 h)^2}{2!} f_{xx} + \frac{(a_{21} h f_i)^2}{2!} f_{yy} + (c_2 h)(a_{21} h f_i) f_{xy}\right] + O(h^4) \\
&\qquad \text{(by Taylor's series expansion of two variables)} \\
y_{i+1} &= y_i + (W_1 + W_2) h f_i + h^2 (W_2 c_2 f_x + W_2 a_{21} f f_y) + \frac{h^3}{2} W_2 (c_2^2 a_{21} f f_{xy} + a_{21}^2 f^2 f_{yy}) + O(h^4)
\end{aligned}
$$

Now by comparing the equal powers of h in y_{i+1} of two equations, we get

$$W_1 + W_2 = 1, \qquad c_2 W_2 = \frac{1}{2} \qquad \text{and} \qquad a_{21} W_2 = \frac{1}{2}.$$

The solution of this system is

$$a_{21} = c_2, \quad W_2 = \frac{1}{2c_2} \qquad \text{and} \qquad W_1 = 1 - \frac{1}{2c_2}.$$

where c_2 is any arbitrary constant not equal to zero.

Runge-Kutta Method of Order 2

Since 2-stage RK method compares with Taylor series upto h^2 for any value of c_2, the 2-stage RK method is of order two and hence this scheme is denoted in many text books as a **second order RK method**.

For these values of a_{21}, W_2, W_1, **there exists an infinite number of** 2-**stage Runge-Kutta methods of order** 2**.**

Now, to give some numerical values to a_{21}, W_2, W_1 ; first the value of c_2 has to be fixed.

Generally the value of c_2 is fixed such that the values of a_{21}, W_2, W_1 are integers or some real numbers which easy to remember.

Two of such cases, the values of c_2 are $\frac{1}{2}$ and 1 respectively.

Runge-Kutta Method of Order 2

Let $c_2 = 1, a_{21} = 2, W_2 = W_1 = \frac{1}{2}$.

The corresponding 2-stage (second order) RK method is

$$\begin{aligned} K_1 &= hf(x_i, y_i) \\ K_2 &= hf(x_i + h, y_i + K_1) \\ y_{i+1} &= y_i + \frac{K_1 + K_2}{2} \quad \text{for} \quad i = 0, 1, 2, \ldots. \end{aligned}$$

or equivalently

$$y_{i+1} = y_i + \frac{h}{2}\Big[f(x_i, y_i) + f(x_i + h, y_i + hf(x_i, y_i))\Big] \quad \text{for} \quad i = 0, 1, 2, \ldots$$

which is nothing but the **Modified Euler's method.**

Butcher Tableau for Runge-Kutta Method of Order 2

Here are three methods based on convenient choices of c_2. Note that the first two methods are due to Runge.

The Butcher tableau for Runge-Kutta method of order 2 are as follows:

$$
\begin{array}{c|cc}
0 & & \\
\frac{1}{2} & \frac{1}{2} & \\
\hline
 & 0 & 1
\end{array}
\qquad
\begin{array}{c|cc}
0 & & \\
1 & 1 & \\
\hline
 & \frac{1}{2} & \frac{1}{2}
\end{array}
\qquad
\begin{array}{c|cc}
0 & & \\
\frac{2}{3} & \frac{2}{3} & \\
\hline
 & \frac{1}{4} & \frac{3}{4}
\end{array}
$$

Runge-Kutta Method of Order 3

The thrid order Runge-Kutta method is given by

$$y_{i+1} = y_i + \frac{1}{6}\Big[K_1 + 4K_2 + K_3\Big] \quad \text{for} \quad i = 0, 1, 2, \ldots,$$

where

$$\begin{aligned} K_1 &= hf(x_i, y_i) \\ K_2 &= hf(x_i + \frac{h}{2}, y_i + \frac{K_1}{2}) \\ K_3 &= hf(x_i + h, y_i - K_1 + 2K_2) \end{aligned}$$

Butcher Tableau for Runge-Kutta Method of Order 3

The Butcher tableau for Runge-Kutta method of order 3 are as follows:

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ 1 & -1 & 2 & \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array} \qquad \begin{array}{c|ccc} 0 & & & \\ 1 & 1 & & \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \\ \hline & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{array} \qquad \begin{array}{c|ccc} 0 & & & \\ \frac{1}{3} & \frac{1}{3} & & \\ 1 & -1 & 2 & \\ \hline & 0 & \frac{3}{4} & \frac{1}{4} \end{array}$$

$$\begin{array}{c|ccc} 0 & & & \\ \frac{2}{3} & \frac{2}{3} & & \\ \frac{2}{3} & 0 & \frac{2}{3} & \\ \hline & \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \end{array} \qquad \begin{array}{c|ccc} 0 & & & \\ \frac{2}{3} & \frac{2}{3} & & \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \\ \hline & \frac{1}{4} & 0 & \frac{3}{4} \end{array} \qquad \begin{array}{c|ccc} 0 & & & \\ \frac{2}{3} & \frac{2}{3} & & \\ 0 & -1 & 1 & \\ \hline & 0 & \frac{3}{4} & \frac{1}{4} \end{array}$$

Runge-Kutta Method of Order 4

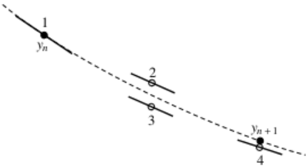
The fourth order Runge-Kutta method is given by

$$y_{i+1} = y_i + \frac{1}{6}\Big[K_1 + 2K_2 + 2K_3 + K_4\Big] \quad \text{for} \quad i = 0, 1, 2, \ldots,$$

where

$$\begin{aligned} K_1 &= hf(x_i, y_i) \\ K_2 &= hf(x_i + \frac{h}{2}, y_i + \frac{K_1}{2}) \\ K_3 &= hf(x_i + \frac{h}{2}, y_i + \frac{K_2}{2}) \\ K_4 &= hf(x_i + h, y_i + K_3) \end{aligned}$$

Runge-Kutta Method of Order 4



In each step of fourth-order Runge-Kutta method, **the derivative is evaluated four times**:

- once at the initial point,
- twice at trial midpoints, and
- once at a trial endpoint.

From these derivatives the final function value y_{n+1} (shown as a filled dot) is calculated.

Butcher Tableau for Runge-Kutta Method of Order 4

The Butcher tableau for Runge-Kutta method of order 4 are as follows:

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array} \qquad \begin{array}{c|cccc} 0 & & & & \\ \frac{1}{4} & \frac{1}{4} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 1 & -2 & 2 & \\ \hline & \frac{1}{6} & 0 & \frac{2}{3} & \frac{1}{6} \end{array} \qquad \begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ 0 & -1 & 1 & & \\ 1 & -1 & \frac{3}{2} & \frac{1}{2} & \\ \hline & \frac{1}{12} & \frac{2}{3} & \frac{1}{12} & \frac{1}{6} \end{array}$$

$$\begin{array}{c|cccc} 0 & & & & \\ 1 & 1 & & & \\ \frac{1}{2} & \frac{3}{8} & \frac{1}{8} & & \\ 1 & -2 & -1 & 4 & \\ \hline & \frac{1}{6} & \frac{1}{12} & \frac{2}{3} & \frac{1}{12} \end{array} \qquad \begin{array}{c|cccc} 0 & & & & \\ \frac{1}{3} & \frac{1}{3} & & & \\ \frac{2}{3} & \frac{-1}{3} & 1 & & \\ 1 & 1 & -1 & 1 & \\ \hline & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{array} \qquad \begin{array}{c|cccc} 0 & & & & \\ \frac{2}{3} & \frac{2}{3} & & & \\ \frac{1}{3} & \frac{1}{12} & \frac{1}{4} & & \\ 1 & -\frac{5}{4} & \frac{1}{4} & 2 & \\ \hline & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{array}$$

Working Rule

For a given point (x_0, y_0) in the (unknown) solution curve, we are going to find the ordinate of given x.

Divide the interval connecting x_0 and x into n subintervals each of width h, the value of h is quite small.

To apply one of the Runge-Kutta methods, we should calculate the weights K_i at (x_0, y_0) to find $y_1 = y(x_0 + h)$.

Similarly, the value of y in the second interval is obtained by replacing x_0 by x_1 and y_0 by y_1 in the Runge-Kutta method and we obtain y_2.

In general, to find y_n substitute x_{n-1}, y_{n-1} in the expression for K_1, K_2 etc.

To evaluate y_{n+1}, we need information only at the point y_n. Information at the points y_{n-1}, y_{n-2} etc. are not directly required . Hence Runge-Kutta methods are **step methods.**

Exercises

1. Compute $y(0.1)$ and $y(0.2)$ by Runge-Kutta method of fourth order for the differential equation
$$\frac{dy}{dx} = xy + y^2, \quad y(0) = 1.$$
2. Use Runge-Kutta method of fourth order to find $y(0.1)$, given that
$$\frac{dy}{dx} = \frac{1}{x+y}, \quad y(0) = 1.$$
3. Given $y' = x^2 - y,\ y(0) = 1$, find $y(0.1)$ using Runge-Kutta method of fourth order.
4. Using 4th order Runge-Kutta method, find $y(0.1), y(0.2)$ and $y(0.3)$, given that
$$\frac{dy}{dx} = 1 + xy, \quad y(0) = 2.$$

Exercises

5. Using 4th order Runge-Kutta method, evaluate the value of y when $x = 1.1$, given that

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}, \quad y(1) = 1.$$

6. Apply third order Runge-Kutta method to find an approximate value of y when $x = 0.2$, given that

$$\frac{dy}{dx} = x + y, \quad y(0) = 1.$$

7. Using Runge-Kutta method of fourth order, solve

$$\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$$

with $y(0) = 1$ at $x = 0.2,\ 0.4$.

Exercises

8. Apply Runge-Kutta method to find an approximate value of y when $x = 0.2$ in steps of 0.1 if
$$\frac{dy}{dx} = x + y^2$$
given that $y = 1$ where $x = 0$.

9. Using Runge-Kutta method of fourth order, solve for y at $x = 1.2, 1.4$ from
$$\frac{dy}{dx} = \frac{2xy + e^x}{x^2 + xe^x}$$
given $x_0 = 1, y_0 = 0$.

10. Given
$$\frac{dy}{dx} = y - x$$
where $y(0) = 2$, find $y(0.1)$ and $y(0.2)$ correct to 4 decimal places by
    - Runge-Kutta second order formula
    - Runge-Kutta fourth order formula.

Exercises

11. Given
$$\frac{dy}{dx} = 1 + y^2$$
where $y(0) = 0$, find $y(0.2), y(0.4)$ and $y(0.6)$ by Runge-Kutta fourth order formula.

12. Taking $h = 0.5$, solve the initial value problem for $x = 0.05$ of the differential equation
$$\frac{dy}{dx} = 3x + \frac{y}{2}, \quad y(0) = 1.$$

    - Euler's method
    - Modified Euler's method
    - Runge-Kutta method of order 4.

Solving Simultaneous First Order Differential Equations : Runge-Kutta Methods

Consider the simultaneous differential equations of the type

$$\frac{dy}{dx} = f(x, y, z) \tag{1}$$

and

$$\frac{dz}{dx} = g(x, y, z) \tag{2}$$

with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ can be solved by Runge-Kutta methods.

Starting at (x_i, y_i, z_i) and taking the step-sizes for x, y, z to be h, k, ℓ respectively, we have the following for the Runge-Kutta methods of fourth order.

$$\begin{aligned}
K_1 &= h\ f(x_i, y_i, z_i) & J_1 &= h\ g(x_i, y_i, z_i) \\
K_2 &= h\ f(x_i + \frac{h}{2}, y_i + \frac{K_1}{2}, z_i + \frac{J_1}{2}) & J_2 &= h\ g(x_i + \frac{h}{2}, y_i + \frac{K_1}{2}, z_i + \frac{J_1}{2}) \\
K_3 &= h\ f(x_i + \frac{h}{2}, y_i + \frac{K_2}{2}, z_i + \frac{J_2}{2}) & J_3 &= h\ g(x_i + \frac{h}{2}, y_i + \frac{K_2}{2}, z_i + \frac{J_2}{2}) \\
K_4 &= h\ f(x_i + h, y_i + K_3, z_i + J_3) & J_4 &= h\ g(x_i + h, y_i + K_3, z_i + J_3)
\end{aligned}$$

$$y_{i+1} = y_i + \frac{1}{6}\Big[K_1 + 2K_2 + 2K_3 + K_4\Big] \text{for} \quad i = 0, 1, 2, \ldots,$$

and

$$z_{i+1} = z_i + \frac{1}{6}\Big[J_1 + 2J_2 + 2J_3 + J_4\Big] \text{for} \quad i = 0, 1, 2, \ldots.$$

Exercises

13. Solve the system of differential equations

$$\frac{dy}{dx} = xz + 1, \qquad \frac{dz}{dx} = -xy$$

for $x = 0.3(0.3)0.9$ using Runge-Kutta fourth order formula. Initial values are $x = 0, y = 0, z = 1$.

14. Using Runge-Kutta method of order 4, find the approximate values of x and y at $t = 0.2$ for the following system:

$$\frac{dx}{dt} = 2x + y, \qquad \frac{dy}{dt} = x - 3y$$

given that when $t = 0$, $x = 0, y = 0.5$.

Solving Second Order Differential Equations by Runge-Kutta Methods

Let us describe the fourth order Runge-Kutta method for solving the second order differential equation

$$\frac{d^2y}{dx^2} = y'' = f(x, y, y'). \tag{3}$$

If we put $y' = z$, then $y'' = z'$. So the differential equation becomes

$$z' = f(x, y, z).$$

We have now the simultaneous differential equations

$$\frac{dy}{dx} = z \quad (\text{that is,} \quad f_1 \equiv z) \quad \text{and} \quad z = \frac{dz}{dx} = f(x, y, z) \quad (\text{that is,} \quad f_2 \equiv f)$$

where f_1 means $f_1(x, y, z)$ and f_2 denotes $f_2(x, y, z)$.

Exercises

15. Given
$$\frac{d^2y}{dx^2} - y^3 = 0, \quad y(0) = 10, \quad y'(0) = 5.$$
Evaluate $y(0.1)$ using Runge-Kutta method.
16. Use the Runge-Kutta method with fourth order accuracy to determine the approximate value of y at $x = 0.1$ if y satisfies the differential equation
$$\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1, \quad y(0) = 1, \quad y'(0) = 0.$$
17. Using Runge-Kutta method, solve
$$\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$$
for $x = 0.2$ correct to 4 decimal places. Initial conditions are $y(0) = 1, y'(0) = 0$.

A predictor-corrector method is an algorithm that proceeds in two steps. First, the prediction step calculates a rough approximation of the desired quantity. Second, the corrector step refines the initial approximation using another means. This involves

- **The predictor step.** We use an explicit method to obtain an approximation y^p_{n+k} to y_{n+k}.
- **The corrector step.** We use an implicit method, but with the predicted value y^p_{n+k} on the right-hand side in the evaluation of f_{n+k}. We use f^p_{n+k} to denote this approximate (predicted) value of f_{n+k}.
- We can then go on to correct again and again. At each step we put the latest approximation to y_{n+k} in the right-hand side of the scheme to generate a new approximation from the left-hand side.

What has been done in Newton-Raphson method?

This is not unlike an implementation of Newton-Raphson. In that method we require an initial guess (we "predict") and then the Newton-Raphson approach tells us how to iterate (or "correct") our latest approximation.

The main difference here is that **we have a systematic way of obtaining the initial prediction.**

It is sufficient for our purposes to illustrate the idea of a predictor-corrector method using the simplest possible pair of methods.

We use Euler's method to predict and the trapezium method to correct.

Predictor-Corrector Methods

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{given} \quad y(x_0) = y_0.$$

We divide the range for x into a number of step sizes of equal width h. If x_i and x_{i+1} are two consecutive points, then $x_{i+1} = x_i + h$.

Euler's formula for the above differential equation

$$y_{i+1} = y_i + hf(x_i, y_i), \quad i = 0, 1, 2, \ldots. \tag{4}$$

The modified Euler's formula is

$$y_{i+1} = y_i + \frac{h}{2}\Big[f(x_i, y_i) + f(x_{i+1}, y_{i+1})\Big], \quad i = 0, 1, 2, \ldots. \tag{5}$$

Predictor-Corrector Methods

The value of y_{i+1} is first estimated by (4) and this value is inserted on the right side of (5) to get a better approximation for y_{i+1}.

This value of y_{i+1} is again substituted in (5) to find a still better approximation of y_{i+1}.

This process is repeated until two consecutive values of y_{i+1} are almost equal.

This technique of refining an initially crude estimate by means of more accurate formula is known as **predictor-corrector methods**. Equation (4) is called the **predictor** and (5) is called **corrector**.

Predictor-Corrector Methods

In the method described so far to solve a differential equation over an interval only the value of y at the beginning of the interval was required.

In the predictor and corrector methods, **four prior values of y are needed** to evaluate the value of y at x_{i+1}.

A **predictor formula** is used to predict the value y_{i+1} of y at x_{i+1} and then a **corrector formula** is used to improve the value of y_{i+1}.

Milne's Method

Consider the first order differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{given} \quad y(x_0) = y_0.$$

Newton's forward difference formula can be written as

$$f(x, y) = f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 f_0 + \cdots . \quad (6)$$

Substituting this in the relation

$$y_4 = y_0 + \int_{x_0}^{x_0+4h} f(x, y)dx$$

we get

$$y_4 = y_0 + \int_{x_0}^{x_0+4h} \left[f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \cdots \right] dx.$$

Milne's Method

Let $x = x_0 + nh$. Therefore

$$\begin{aligned}
y_4 &= y_0 + h\int_0^4 \left[f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \cdots\right] dn \\
&= y_0 + h\left[4f_0 + 8\Delta f_0 + \frac{20}{3}\Delta^2 f_0 + \frac{8}{3}\Delta^3 f_0 + \cdots\right] \\
&= y_0 + h\left[4y_0' + 8(E-1)y_0' + \frac{20}{3}(E^2 - 2E + 1)y_0' + \frac{8}{3}(E^3 - 3E^2 + 3E - 1)y_0'\right] \\
&\qquad \text{(neglecting fourth and higher order differences)} \\
&= y_0 + h\left[4y_0' + 8(y_1' - y_0') + \frac{20}{3}(y_2' - 2y_1' + y_0') + \frac{8}{3}(y_3' - 3y_2' + 3y_1' - y_0')\right] \\
&= y_0 + h\left[\frac{8}{3}y_1' - \frac{4}{3}y_2' + \frac{8}{3}y_3'\right].
\end{aligned}$$

Milne's Method

Thus

$$y_4 = y_0 + \frac{4h}{3}\left[2y'_1 - y'_2 + 2y'_3\right].$$

If $x_0, x_1, \ldots, x_4$ are any 5 consecutive values of x, then the above equation can be written as

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right].$$

This is called **Milne's Predictor formula** (the subscript p indicates that it is a predicted value).

This formula can be used to predict the value of y_4 when those of y_0, y_1, y_2, y_3 are known.

Milne's Method

To get a corrector formula we substitute Newton's formula (6) in the relation

$$y_2 = y_0 + \int_{x_0}^{x_0+2h} f(x,y)dx$$

and we get

$$\begin{aligned}
y_2 &= y_0 + \int_{x_0}^{x_0+2h} \left[f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \cdots \right] dx \\
&= y_0 + h\int_0^2 \left[f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \cdots \right] dn \ \text{ putting } x = x_0 + nh \\
&= y_0 + h\left[2f_0 + 2\Delta f_0 + \frac{1}{3}\Delta^2 f_0 + \cdots \right] \\
&= y_0 + h\left[2y_0' + 2(E-1)y_0' + \frac{1}{3}(E^2 - 2E + 1)y_0' \right] \\
&\qquad \text{(neglecting higher order differences)} \\
&= y_0 + h\left[2y_0' + 2(y_1' - y_0') + \frac{1}{3}(y_2' - 2y_1' + y_0') \right]
\end{aligned}$$

Milne's Method

Thus

$$y_2 = y_0 + \frac{h}{3}\left[y_0' + 4y_1' + y_2'\right].$$

If x_0, x_1, x_2 are any three corrective values of x, the above relation can be written as

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y_{n-1}' + 4y_n' + y_{n+1}'\right]$$

This is known **Milne's corrector formula** where the suffix c stands for corrector.

An improved value of y_{n+1}' is computed and again the corrector formula is applied until we get y_{n+1} to the derived accuracy.

Adams-Bashforth Method

Consider

$$\frac{dy}{dx} = f(x, y) \qquad \text{with} \quad y(x_0) = y_0.$$

Newton's backward interpolation formula can be written as

$$f(x, y) = f_0 + n\nabla f_0 + \frac{n(n+1)}{2!}\nabla^2 f_0 + \frac{n(n+1)(n+2)}{6}\nabla^3 f_0 + \cdots$$

where $n = \frac{x - x_n}{h}$ and $f_0 = f(x_0, y_0)$.

Adams-Bashforth Method

Substituting this

$$y_1 = y_0 + \int_{x_0}^{x_0+h} f(x,y)dx, \tag{7}$$

we get

$$\begin{aligned} y_1 &= y_0 + \int_{x_0}^{x_0+h} \left[f_0 + n\nabla f_0 + \frac{n(n+1)}{2!}\nabla^2 f_0 + \cdots \right] dx \\ &= y_0 + \int_0^1 \left[f_0 + n\nabla f_0 + \frac{n(n+1)}{2!}\nabla^2 f_0 + \cdots \right] dn \qquad (\text{putting} \;\; x = x_0 + nh) \\ &= y_0 + h\left[f_0 + \frac{1}{2}\nabla f_0 + \frac{5}{12}\nabla^2 f_0 + \frac{3}{8}\nabla^3 f_0 + \cdots \right] \end{aligned}$$

Neglecting fourth and higher order differences and expressing ∇f_0, $\nabla^2 f_0$, $\nabla^3 f_0$ in terms of function values we get

$$y_1 = y_0 + \frac{h}{24}\left[55y_0' - 59y_{-1}' + 37y_{-2}' - 9y_{-3}' \right].$$

Adams-Bashforth Formula

This can also be written as

$$y_{n+1,p} = y_0 + \frac{h}{4}\left[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}\right].$$

This is called **Adams-Bashforth formula** and is used as a **predictor formula**. The notation p indicates that it is a predicted value.

A corrector formula can be derived in a similar manner by using Newton's backward difference formula at f_1.

That is,

$$f(x,y) = f_1 + n\nabla f_1 + \frac{n(n+1)}{2!}\nabla^2 f_1 + \frac{n(n+1)(n+2)}{3!}\nabla^3 f_1 + \cdots$$

Adams-Moulton Corrector Formula

Substituting this in (7), we get,

$$\begin{aligned} y_1 &= y_0 + \int_{x_0}^{x_1} \left[f_1 + n\nabla f_1 + \frac{n(n+1)}{2!}\nabla^2 f_1 + \cdots \right] dx \\ &= y_0 + h\int_{-1}^{0} \left[f_1 + n\nabla f_1 + \frac{n(n+1)}{2!}\nabla^2 f_1 + \cdots \right] dn \qquad (\text{putting } \; x = x_1 + nh) \\ &= y_0 + h\left[f_1 - \frac{1}{2}\nabla f_1 - \frac{1}{12}\nabla^2 f_1 + \frac{1}{24}\nabla^3 f_1 - \cdots \right] \end{aligned}$$

Neglecting 4th and higher order differences and expressing ∇f_1, $\nabla^2 f_1$, $\nabla^3 f_1$ in terms of function values, we get

$$y_1 = y_0 + \frac{h}{24}\Big[9f_1 + 19f_0 - 5f_{-1} + f_{-2} \Big].$$

Adams-Moulton Corrector Formula

This can also be written as

$$y_{n+1,c} = y_n + \frac{h}{24}\left[9f_{n+1}^{p} + 19f_n - 5f_{n-1} + f_{n-2}\right].$$

This is called **Adams-Moulton corrector formula**.

The subscript c indicates that the value obtained in the corrected value and the superscripts p on the right indicates that the predicted value of y_{n+1} should be used for computing the value of $f(x_{n-1}, y_{n-1})$.

Important Note

To apply a predictor-corrector method, we need four starting values of y which can be calculated by any of the numerical methods.

In practice, the 4th order Runge-Kutta method is found to be most useful. Once the predictor-corrector method starts, it is very effective, as past information is effectively used.

To apply Milne's predictor-corrector method, we require 4 prior values of y.

If these values are not given, we can determine them by using Taylor's series, Euler's or Runge-Kutta methods.

Exercises

18. Given
$$\frac{dy}{dx} = \frac{1}{x+y}, \quad y(0) = 2.$$
If $y(0.2) = 2.09, y(0.4) = 2.17$ and $y(0.6) = 2.24$, find $y(0.8)$ using Milne's method.

19. Using Milne's predictor-corrector formula, find $y(0.4)$, for the differential equation
$$\frac{dy}{dx} = 1 + xy, \quad y(0) = 2.$$

20. Given
$$\frac{dy}{dx} = \frac{1}{2}(1+x^2)y^2$$
and $y(0) = 1, y(0.1) = 1.6, y(0.2) = 1.12, y(0.3) = 1.21$. Evaluate $y(0.4)$ by Milne's predictor-corrector method.

21. Find by Milne's method for the equation

$$y' = y - x^2, \quad y(0) = 1,$$

by obtaining the starting values by Taylor's series method.

22. Using Adams-Bashforth method find $y(4.4)$ given

$$5xy' + y^2 = 2, \quad y(4) = 1, \quad y(4.1) = 1.0049, \quad y(4.2) = 1.0097$$

and $y(4.3) = 1.0143$.

23. Using Adams-Bashforth method determine $y(1.4)$ given that

$$y' - x^2y = x^2, \quad y(1) = 1.$$

Obtain the starting values from the Euler's method.

24. Using Adams-Bashforth method find $y(0.4)$ given that
$$y' = 1 + xy, \quad y(0) = 2.$$

25. Apply Milne's method to find $y(0.4)$ of the initial value problem
$$y' = x - y^2, \quad y(0) = 1.$$
Starting solutions required are to be obtained using Runge-Kutta method of order 4 using step value $h = 0.1$.

26. Using Milne's method, find $y(4.5)$ given
$$5xy' + y^2 - 2 = 0$$
with $y(4) = 1, y(4.1) = 1.0049, y(4.2) = 1.0097$, $y(4.3) = 1.0143$, $y(4.4) = 1.0187$.

Exercises

27. Given
$$y' = x(x^2 + y^2)e^{-x}, \quad y(0) = 1,$$
find y at $x = 0.1, 0.2$, and 0.3 by Taylor's series method and compute $y(0.4)$ by Milne's method.

28. Using Runge-Kutta method of order 4, find y for $x = 0.1, 0.2, 0.3$ given that
$$\frac{dy}{dx} = xy + y^2, \quad y(0) = 1.$$
Continue the solution at $x = 0.4$ using Milne's method.

29. If
$$\frac{dy}{dx} = 2e^x y, \quad y(0) = 2,$$
find $y(4)$ using Adams predictor-corrector formula by calculating $y(1), y(2)$ and $y(3)$ using Euler's modified formula.

30. Given
$$y'' + xy' + y = 0, \quad y(0) = 1, \quad y'(0) = 0$$
obtain y for $x = 0.1, 0.3$ by any method. Further, continue the solution by Milne's method to calculate $y(0.4)$.

31. Given
$$\frac{dy}{dx} = 1 + y^2$$
where $y = 0$ when $x = 0$. Find $y(0.8)$ by Adams-Bashforth formula. Find $y(0.2), y(0.4), y(0.6)$ by fourth order Runge-Kutta-method.

32. Given
$$\frac{dy}{dx} = 1 + y^2$$
where $y = 0$ when $x = 0$. Find $y(0.8)$ and $y(1.0)$ by Milne's formula.

Exercises

33. The differential equation $y' = x^2 + y^2 - 2$ satisfies the following data:

x	y
-0.1	1.0900
0	1.0000
0.1	0.8900
0.2	0.7605

Use Milne's method to obtain the value of $y(0.3)$.

34. Using Adams-Bashforth predictor-corrector formulae, evaluate $y(1.4)$ if y satisfies

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$$

and $y(1) = 1, y(1.1) = 0.996, y(1.2) = 0.986, y(1.3) = 0.972$.

Exercises

35. Find $y(2)$ by Milne's method if $y(x)$ is the solution of
$$\frac{dy}{dx} = \frac{1}{2}(x+y)$$
assuming $y(0) = 2, y(0.5) = 2.636, y(1) = 3.595, y(1.5) = 4.968.$

36. Tabulate by Milne's method the numerical solution of
$$\frac{dy}{dx} = x + y$$
from $x = 0.2$ to $x = 0.3$ given that

x	y
0	1
0.05	1.0526
0.1	1.1104
0.15	1.1737

37. Solve the differential equation

$$\frac{dy}{dx} = x^2 + y^2 - 2$$

given $y(-0.1) = 1.09, y(0) = 1, y(0.1) = 0.89$. Find $y(0.2)$ by series expansion and find $y(0.3)$ by Milne's method.

38. Solve the initial value problem

$$\frac{dy}{dx} = 1 + xy^2, \quad y(0) = 1$$

for $x = 0.4, 0.5$ by using Milne's method when it is given that

x	0.1	0.2	0.3
y	1.105	1.223	1.355

39. Using the Adams method, solve the simultaneous differential equations

$$\begin{aligned} \frac{dy}{dx} &= xy + z \\ \frac{dz}{dx} &= y - z \end{aligned}$$

with $y(0) = 0, \ z(0) = 1$.

40. Use Milne's method to solve the simultaneous differential equations

$$\begin{aligned} \frac{dy}{dx} &= x + z \\ \frac{dz}{dx} &= -xy \end{aligned}$$

with $y(0) = 1, \ z(0) = 0$.

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